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*- Products in the method of orbits for nilpotent groups

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Abstract. *On each orbit W of the coadjoint representation of any nilpotent (connected, simply connected) Lie group G, we construct *-products and associated Von Neumann algebras $\mathfrak{G}$. G acts canonically on $\mathfrak{G}$ by automorphisms. In the unique faithful, irreducible representation of $\mathfrak{G}$, this action is implemented by the unitary irreducible representation of G corresponding to W by the Kirillov method. This construction is uniquely determined by W and gives the classification of all unitary irreducible representations of G.*

INTRODUCTION

Deformations of Poisson brackets and associative algebras of C^∞-functions on symplectic manifolds (classical phase-spaces) permit an autonomous quantization theory without need for Hilbert space operators. This theory is based on the notion of *-products introduced by Flato and Lichnerowicz (see for instance [1]). This quantization differs essentially from the geometrical quantization initiated by Kirillov, Kostant and Souriau ([2]).

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Let us consider now the problem of construction and classification of unitary irreducible representations (U.I.R.) of a Lie group G. The so-called «method of orbits» uses the orbits of the coadjoint representation of G which are symplectic manifolds with natural action of G. This method was initiated for nilpotent groups by Kirillov ([3]) and the result is an one to one correspondence between orbits and UIR's of G. Moreover the geometrical quantization on each orbit is unique (up to an isomorphism) and gives an Hilbertian realization of the corresponding UIR.

In quantization by $*$-products, some realizations of groups representations naturally happen (see [4] for various examples). A first stage for a systematical theory of $*$-products on orbits of the coadjoint representation was performed in [5] and [6].

Concerning nilpotent connected and simply connected Lie groups, $*$-products on each coadjoint orbit of G were defined. They are formal deformations, invariant under a formal representation of G. Using a polarization, an intertwining operator is defined on a suitable subspace of convergent series between the UIR corresponding to the orbit and a formal representation of G.

This formal approach privilegies the parametrization defining the formal $*$-product; thus two distincts $*$-products «converge» on two distinct subspaces of the UIR realization and there is no global intertwining between the associated formal representations.

In this article we adopt directly a functional description of $*$-products by integral formulas on a canonical space of C^∞-functions on the orbit. We realize this $*$-algebra of functions as operators on the L^2-space of the orbit; its closure $\mathfrak{G}$ is a Von Neumann algebra. We prove that G acts by automorphisms of $\mathfrak{G}$ and give explicitely the unique faithful irreducible representation of $\mathfrak{G}$ on a Hilbert space. In this representation the action of G on $\mathfrak{G}$ is unitarily implemented by the UIR of G corresponding to the orbit. This construction is independent (up to an unitary equivalence) of the parametrization of the orbit.

1. $*$-PRODUCTS

1.1. Formal $*$-products

Let W be a connected paracompact symplectic differentiable manifold.

DEFINITION 1.1. A (formal) *$*$-product* on W is a deformation $*_\lambda$ of the associative algebra $C^\infty(W)$ in the space of formal series with parameter λ:

$$(1) \qquad u *_\lambda v = u \cdot v + \sum_{r \geq 1} (-\lambda)^r C^r(u, v); \quad u, v \in C^\infty(W),$$

sucht that:

i) C^r is a bidifferential operator, vanishing on constants

ii) $C^1(u, v) = \{u, v\}$(Poisson bracket of u and v)

iii) $C^r(u, v) = (-1)^r C^r(v, u)$.

We denote by $[\,.\,,\,.\,]_\lambda$ the associated deformation of the Lie algebra $(C^\infty(W), \{\,.\,,.\})$:

(2) $$[u, v]_\lambda = -\frac{1}{2\lambda}(u *_\lambda v - v *_\lambda u); \quad u, v \in C^\infty(W).$$ ■

We shall be concerned further by *-products on orbits of the coadjoint representation of a nilpotent Lie group. Therefore we look for eventual invariance property of *-products on a group action on W.

DEFINITION 1.2. Let G be a group acting by symplectomorphisms on W. A *-product $*_\lambda$ is called *G-invariant* iff:

$$\forall r > 0, \; C^r(g \cdot u, g \cdot v) = g \cdot C^r(u, v); \quad u, v \in C^\infty(W); \quad g \in G,$$

where $g.$ is the natural action of G on $C^\infty(W)$:

(3) $$(g \cdot u)(\xi) = u(g^{-1} \cdot \xi); \; g \in G; \; u \in C^\infty(W); \; \xi \in W.$$ ■

If it is quite natural to envisage this invariance property, the following proposition proves that it is not relevant for our purpose here.

PROPOSITION 1.1. ([6]). *Let g be the nilpotent Lie algebra with basis $(X_0, X_1, \ldots, X_n, Y)$ $(n \geqslant 3)$ with only nonvanishing Lie brackets: $[Y, X_i] = X_{i-1}$ $(i = 1, 2, \ldots, n)$. On the generic orbits of the coadjoint representation of the connected and simply connected corresponding Lie group G there exists no G-invariant *-products.* ■

This negative result justifies the consideration of the following weaker invariance condition:

DEFINITION 1.3. Let G be a group acting by symplectomorphisms on W. A *-products $*_\lambda$ is called *G-covariant* if there exists a representation of G by *-automorphisms which is a deformation of the «geometrical» representation of G given by (3):

$$\tau_g = \left(\mathrm{Id} + \sum_{r>0} \lambda^r a_g^r\right) \circ g.; \qquad g \in G,$$

where the a^r's are differential operators vanishing on constants and:

$$\left.\begin{array}{l}\tau_{gg'} = \tau_g \circ \tau_{g'} \\ \\ \tau_g(u *_\lambda v) = \tau_g(u) *_\lambda \tau_g(v)\end{array}\right\} g, g' \in G; \quad u, v \in C^\infty(W).$$ ■

This notion was introduced in [6] and its interest lies in the following result:

PROPOSITION 1.2. ([5]). *Let G be a connected simply connected Lie group with Lie algebra g, acting by symplectomorphisms on W.*

For each $X \in g$, we denote by X^- the vector field on W defined by:

$$(4) \qquad (X^- u)(\xi) = \frac{d}{dt}\, u(\exp(-tX \cdot \xi)|_{t=0}; \quad \xi \in W, \ u \in C^\infty(W).$$

Let us suppose that:

i) *the vector fields X^- are globally hamiltonian*

ii) *there exists an hamiltonian function $\widetilde{X}$ for each X^- such that*

$$\{\widetilde{X}, \widetilde{Y}\} = \widetilde{[X, Y]}.$$

Then, each $*$-product satisfying to:

$$(5) \qquad [\widetilde{X}, \widetilde{Y}]_\lambda = \{\widetilde{X}, \widetilde{Y}\}; \quad X, Y \in g$$

is covariant with respect to G. ■

1.2. Integral formulas on $\mathbb{R}^{2k}$

Let (p_j, q_j) $(j = 1, 2, \ldots, k)$ be a canonical coordinates system for $\mathbb{R}^{2k}$. We define a symplectic 2-form σ on $\mathbb{R}^k$ by:

$$\sigma = \sum_{j=1}^{k} dp_j \wedge dq_j.$$

Let us denote by $\partial_j u$, $\frac{\partial u}{\partial p_j}$ if $j \leqslant k$, $\frac{\partial u}{\partial q_{j-k}}$ if $k < j \leqslant 2k$. The Poisson bracket of u and v in $C^\infty(W)$ is:

$$\{u, v\} = \sum_{j=1}^{k} \left(\frac{\partial u}{\partial p_j}\frac{\partial u}{\partial q_j} - \frac{\partial u}{\partial q_j}\frac{\partial u}{\partial p_j}\right) = \Lambda^{ij}\, \partial_i u\, \partial_j v.$$

We put

(6) $$P^r(u,v) = \Lambda^{i_1 j_1} \dots \Lambda^{i_r j_r} \partial_{i_1 i_2 \dots i_r} u \, \partial_{j_1 j_2 \dots j_r} v.$$

DEFINITION 1.4. The formula

(7) $$u *_\lambda v = uv + \sum_{r>0} \frac{(-\lambda)^r}{r!} \, p^r(u,v); \quad u, v \in C^\infty(\mathbb{R}^{2k})$$

defines a *-product on $\mathbb{R}^{2k}$. This *-product is called the Moyal *-product on $\mathbb{R}^{2k}$. ■

We shall envisage the convergence of (6) to a product defined by an integral formula in suitable functionnal spaces. We consider first a such product in the Schwartz space $\mathscr{S}(\mathbb{R}^{2k})$.

PROPOSITION 1.3. ([7]). i) *The symplectic Fourier transform F defined on $\mathscr{S}(\mathbb{R}^{2k})$ by*

(8) $$(Fu)(\xi) = (2\pi)^{-k} \int e^{i\sigma(\xi,\xi')} u(\xi') \, d\xi' \quad (d\xi' = dp \, dq)$$

is an involutive isometry of $\mathscr{S}(\mathbb{R}^{2k})$.

ii) *The symplectic convolution product x_σ defined on $\mathscr{S}(\mathbb{R}^{2k})$ by:*

(9) $$(u \, x_\sigma \, v)(\xi) = (2\pi)^{-k} \int e^{-i\sigma(\xi,\xi')} u(\xi') \, v(\xi - \xi') \, d\xi'$$

provides $\mathscr{S}(\mathbb{R}^{2k})$ with an associative algebra structure.

iii) *Let us denote by*

(10) $$u * v = F(F(u) \, x_\sigma \, F(v)) \quad (u, v \in \mathscr{S}(\mathbb{R}^{2k}).$$

Then,

(11) $$(u * v)(\xi) = (2\pi)^{-2k} \iint u(\xi') v(\xi'') e^{i(\sigma(\xi,\xi'') + \sigma(\xi'',\xi') + \sigma(\xi',\xi))} d\xi' \, d\xi''. \; ■$$

Let us call Weyl-Moyal product the product defined by (11) on $\mathscr{S}(\mathbb{R}^{2k})$. An easy computation based on formulas (8) and (9) permits to prove the following identities:

LEMMA 1.1. *Let $u, v \in \mathscr{S}(\mathbb{R}^{2k})$. Then for $j = 1, 2, \ldots, k$,*

$$F\left(\frac{\partial u}{\partial q_j}\right) = -p_j F(u); \qquad F\left(\frac{\partial u}{\partial p_j}\right) = i\, q_j F(u)$$

$$F(p_j u) = i \frac{\partial}{\partial q_j} F(u); \qquad F(q_j u) = -i \frac{\partial}{\partial p_j} F(u)$$

$$\frac{\partial}{\partial q_j}(u_{x_\sigma} v) = u_{x_\sigma} \frac{\partial v}{\partial q_j} + i(p_j u_{x_\sigma} v)$$

$$\frac{\partial}{\partial p_j}(u_{x_\sigma} v) = u_{x_\sigma} \frac{\partial}{\partial p_j} - i(q_j u_{x_\sigma} v)$$

$$q_j(u_{x_\sigma} v) = i \frac{\partial u}{\partial p_j}{}_{x_\sigma} v - i\, u_{x_\sigma} \frac{\partial v}{\partial p_j}$$

$$p_j(u_{x_\sigma} v) = -i \frac{\partial u}{\partial q_j}{}_{x_\sigma} v + i\, u_{x_\sigma} \frac{\partial v}{\partial q_j}. \qquad \blacksquare$$

In fact (11) is an integral formula for the *-Moyal product:
Let (x_j) $(j = 1, 2, \ldots, 2k)$ be the dual basis of (p_j, q_j) i.e.:

$$\langle (x_j), (p_j, q_j) \rangle = \sum_{j=1}^{k} (x_j\, p_j + x_{j+k} q_j).$$

PROPOSITION 1.4. ([8]). *Let u and v be distributions in $\mathscr{S}'(\mathbb{R}^{2k})$ with (usual) Fourier transofrms $\hat{u}$ and $\hat{v}$ with compact support. By the formula:*

$$\langle \hat{u} \circ \hat{v}, \varphi \rangle = \langle \hat{u}(x) \quad \hat{v}(y), e^{-i\frac{\hbar}{2}\Lambda(x,y)} \varphi(x+y) \rangle; \quad \varphi \in C^\infty(\mathbb{R}^{2k})$$

with $\hbar \in \mathbb{R}$ and $\Lambda(x, y) = \Lambda^{ij} x_i y_j$, we define a distribution $\hat{u} \circ \hat{v}$ in $\mathscr{E}'(\mathbb{R}^{2k})$ with compact support. Moreover:

i) *the serie* $u \cdot v + \sum_{r>0} \left(-\frac{i\hbar}{2}\right)^r \frac{1}{r!} P^r(u, v)$ *converges in $\mathscr{S}'(\mathbb{R}^{2k})$ to an element denoted by $u \,\square\, v$.*

ii) $\widehat{u \,\square\, v} = \hat{u} \circ \hat{v}$

iii) *If moreover u and v are in $\mathscr{S}(\mathbb{R}^{2k})$ and $\hbar = 2$, then;*

$$u \,\square\, v = u * v \qquad (\text{* defined by } (10)). \qquad \blacksquare$$

Proof. i) and ii) are proved in [8]. We give here a short version of this proof: For φ in $C^\infty(\mathbb{R}^{2k})$ let us write:

$$R_s(x,y) = \varphi(x+y) \sum_{r>s} \frac{1}{r!}\left(i\frac{\hbar}{2}\Lambda(x,y)\right)^r \qquad (x,y\in\mathbb{R}^{2k})$$

and for $\alpha = (\alpha_1, \ldots, \alpha_k) \in \mathbb{N}^k$:

$$D^\alpha = \frac{\partial|\alpha|}{\partial x_1^{\alpha_1}\ldots\partial x_k^{\alpha_k}} \qquad (|\alpha| = \alpha_1+\alpha_2+\ldots+\alpha_k).$$

Then,

$$D^\alpha R_s(x,y) = \sum_{|\beta|+t<r} (D^\beta\varphi)(x+y)P_t(x,y) \sum_{r>s-t} \frac{1}{r!}\left(i\frac{\hbar}{2}\Lambda(x,y)\right)^r$$

where P_t is polynomial with degree $\leqslant t$.

$D^\alpha R_s$ converges to 0 an any compact set when $s\to+\infty$. Then we can write:

$$\langle \hat{u}\circ\hat{v}, \varphi\rangle = \sum_{r=0}^{\infty} \frac{1}{r!}\left(\frac{i\hbar}{2}\right)^r \langle \hat{u}(x)\otimes\hat{u}(y), (\Lambda(x,y))^r\,\varphi(x+y)\rangle.$$

This last series is convergent in $\mathscr{E}'(\mathbb{R}^{2k})$ and therefore in $\mathscr{S}'(\mathbb{R}^{2k})$. Its sum is:

$$\hat{u}\circ\hat{v} = \sum_{r=0}^{\infty}\frac{1}{r!}\left(i\frac{\hbar}{2}\right)^r \Lambda^{i_1j_1}\ldots\Lambda^{i_rj_r}(x_{i_1}\ldots x_{i_r}u)_{x}\,(y_{j_1}\ldots y_{j_r}v)$$

($_{x_\sigma}$ denotes here the usual convolution product).

i and ii are then obtained by Fourier transform.

If moreover $u\in\mathscr{S}(\mathbb{R}^{2k})$, using the isomorphisms μ (of $\mathbb{R}^{2k}$) and Δ (of $\mathscr{S}(\mathbb{R}^{2k})$) defined by:

$$\langle\mu(\xi),\eta\rangle = -\sigma(\xi,\eta) \quad \text{and} \quad \hat{u}(-\mu(\xi)) = (\Delta\hat{u})(\xi) \quad (\xi,\eta\in\mathbb{R}^{2k}),$$

we verify directly that, for $\hbar = 2$:

$$(\Delta\hat{u})_{x_\sigma}(\Delta\hat{v}) = \Delta(\hat{u}\circ\hat{v})$$

which is equivalent to iii. ∎

Convention. Untill now, we use * (without indice λ) for the product defined in proposition 1.3 with $\hbar = 2$.

PROPOSITION 1.5. ([7]). *If u and $v\in\mathscr{S}(\mathbb{R}^{2k})$ then,*

i) $\overline{u} * \overline{v} = \overline{v * u}$

ii) $\int (u * v)(\xi)\, d\xi = \int uv\, d\xi$

iii) *the linear operator* l_u *defined from* $\mathscr{S}(\mathbb{R}^{2k})$ *into* $\mathscr{S}(\mathbb{R}^{2k})$ *by*

$$l_u(v) = u * v$$

is continuous in $L^2(\mathbb{R}^{2k}, d\xi)$ *and then can be extended to a bounded linear operator (still denoted by* l_u*) on* $L^2(\mathbb{R}^{2k}, d\xi)$. ■

The set $l = \{l_u / u \in \mathscr{S}(\mathbb{R}^{2k})\}$ is an involutive algebra i.e.

$$l_u^* = l_{\overline{u}}; \quad l_u \circ l_v = l_{u*v}.$$

Moreover it was proved in [7] that $(\mathscr{S}(\mathbb{R}^{2k}), *)$ is a generalized Hilbert algebra in $L^2(\mathbb{R}^{2k}, d\xi)$.

Therefore it is natural to consider the associated Von Neumann algebra $\mathfrak{G}$ (p, q) which is the bicommutant of $l_{\mathscr{S}}$ in $\mathscr{L}(L^2(\mathbb{R}^{2k}, dp\, dq)$. The usual Weyl operators $W(\eta)$ ($\eta \in \mathbb{R}^{2k}$) defined by:

$$(W(\eta)u)(\xi) = e^{i\sigma(\xi,\eta)} u(\xi - \eta) \quad (\xi \in \mathbb{R}^{2k}, u \in L^2(\mathbb{R}^{2k}, d\xi)$$

generate $\mathfrak{G}$ (p, q) ([7]). From this fact, it results the following properties of $\mathfrak{G}$ (p, q).

PROPOSITION 1.6. *The map* $l : u \in \mathscr{S}(\mathbb{R}^{2k}) \to l_u \in \mathfrak{G}(p, q)$ *is injective.* $\mathfrak{G}(p, q) = l_{\mathscr{S}}''$ *is a type I factor, strong closure of* $l_{\mathscr{S}}$ *in* $\mathscr{L}(L^2(\mathbb{R}^{2k}, d\xi))$. ■

Proof. Let u be an element of $\mathscr{S}(\mathbb{R}^{2k})$ with compact support in $\prod_{j=1}^{2k} [-a, +a]$ $(a > 0)$. If $u_n \in \mathscr{S}(\mathbb{R}^{2k})$ satisfies to $u_n(\xi) = 1$ if $\xi \in \prod_{j=1}^{2k} [-n, +n]$ and $|u_n(\xi)| < 1$, then $l_{u_n}(f)(\xi) = f(\xi)$ for any $\xi \in \prod_{j=1}^{2k} [a - n, n - a]$.

But $\| l_{u_n}(f) \|_{L^2}^2 = \int (u_n * f)(\overline{u_n * f})\, dp\, dq = \int |u_n f|^2(\xi)\, dp\, dq < \|f\|_{L^2}^2$.

Therefore $|(u_n * f - f, g)| < 2 \|f\|_{L^2} \|g - g_K\|_{L^2} + |(u_n * f - f, g_K)|$ where g_K has a compact support i.e. $u_n * f$ weakly converges to $f \cdot \{l_u f / u \in \mathscr{S}(\mathbb{R}^{2k}), f \in L^2\}$ is dense in $L^2(\mathbb{R}^{2k}, dp\, dq)$ and from ([9]. A.12, p. 334) we conclude that $\mathfrak{G}$ (p, q) is the strong closure of $l_{\mathscr{S}}$.

On the other hand, for $v \in \mathscr{S}(\mathbb{R}^{2k})$, $l_v(u_n)$ converges to v when $n \to +\infty$ which proves injectivity of l.

$\mathfrak{G}$ (p, q) is the Von Neumann algebra generated by the Weyl operators $W(\xi)$ and then is a type I factor ([7]).

More precisely, $(l_u(v))(\xi) = \frac{1}{2^k} \int (Fu)(-\xi')(W(\xi')f)(\xi)\, d\xi'$.

Up to an unitary equivalence $\mathfrak{G}$ (p, q) admits only one irreducible faithful representation. The proof on this last assertion (which is essentially the unicity theorem of Von Neumann) lies in [9]. ■

PROPOSITION 1.7. ([10]). *The Schrödinger representation of* $\mathfrak{G}$ (p, q) *is faithful and irreducible. Any representation of* $\mathfrak{G}$ (p, q) *is a direct sum of multiple of the last one and of a (nonfaithful) representation* ρ *such that*

$$\rho(l_u) = 0 \quad \textit{for} \quad u = e^{-\frac{p^2+q^2}{2}}.$$ ■

In §3 we shall give an explicit realization of the Schrödinger representation.

2. ORBITS IN g*

Let G be a Lie group with the algebra g. G acts in the dual space g^* of g by the coadjoint representation:

(12) $$\langle g \cdot \xi, X\rangle = \langle \xi, Ad\, g^{-1}(X)\rangle; \quad X \in g,\ g \in G,\ \xi \in g^*.$$

Let W be an orbit of G in g^*. Then by differentiation, we define a representation of g in the Lie algebra of complete vector fields on W: to $X \in g$ we associate the vector field X^- defined on W by

(13) $$(X^- u)(\xi) = \frac{\mathrm{d}}{\mathrm{d}t}\, u(\exp - tX \cdot \xi)\big|_{t=0}; \quad u \in C^\infty(W).$$

The tangent space $T_\xi W$ to W at ξ is generated by $\{X_\xi^-, X \in g\}$ and the 2-form σ defined on W by:

(14) $$\sigma_\xi(X_\xi^-, Y_\xi^-) = \langle \xi, [X, Y]\rangle; \quad X, Y \in g,$$

is a symplectic form on W ([3]).

G acts by symplectomorphisms on W and each element $X \in g$ appears as a C^∞ function $\widetilde{X}$ on W:

(15) $$\widetilde{X}(\xi) = \langle \xi, X\rangle; \quad \xi \in W.$$

$\widetilde{X}$ is the hamiltonian function associated to X^- i.e.:

(16) $$X^- u = \{\widetilde{X}, u\}; \quad u \in C^\infty(W)$$

The map $X \in g \to \widetilde{X} \in C^\infty(W)$ is a linear representation of g in the Poisson algebra $(C^\infty(W), \{\ .\ ,\ .\ \})$.

2.1. Parametrization of an orbit

PROPOSITION 2.1. ([6]). *Let G be a connected simply connected, nilpotent Lie group with Lie algebra g. Each orbit W (in g^*) of the coadjoint representation of G admits a global chart C:*

$$C : \xi \in W \to (p, q) \in \mathbb{R}^{2k}$$

such that:

i) *the canonical symplectic form* (14) *is* $\sigma = \mathrm{d}p \wedge \mathrm{d}q = \sum_{i=1}^{k} \mathrm{d}p_i \wedge \mathrm{d}q_i$

ii) *each function* $\widetilde{X}$ $(X \in g)$ *has the form:*

$$\widetilde{X} \circ C^{-1}(p, q) = \sum_{i=1}^{k} \alpha_i(q)\, p_i + \alpha_0(q) \tag{17}$$

where the α_i*'s* $(i \geqslant 0)$ *are polynomial functions in variables* $(q_{i+1}, \ldots, q_k)$. ∎

DEFINITION 2.1. Each chart C on W which satisfy i and ii of proposition 2.1 is called an *adapted chart*. ∎

Each adapted chart carries the $*$-Moyal product (7) from $\mathbb{R}^{2k}$ on W. It is easy to verify the hypothesis of proposition 1.2 for this $*$-product. In particular $[\widetilde{X}, \widetilde{Y}]_\lambda = \{\widetilde{X}, \widetilde{Y}\}; X, Y \in g$.

Therefore to each adapted chart C we associate a G-covariant $*$-product. The corresponding representation τ of G by $*$-automorphisms is obtained by integration of the representation D_λ of g by $*$-derivations:

$$D_\lambda(X)u = [\widetilde{X}, u]_\lambda\,; \quad u \in C^\infty(W).$$

As proved in [5] this last statement allows a theory of formal representations of G in which are all unitary irreducible representations. Generally there is no equivalence between two representations obtained from two distinct adapted charts. In the formal treatment the variables (q_i) play a privilegied role but their choice is non canonical.

We shall prove that the use of the $*$-Weyl Moyal product (11) on an suitable functional space defines a canonical construction of the unitary irreducible representation in analogy to the Kirillov theory.

2.2. The space $\mathcal{S}(W)$

Given an adapted chart (p, q) on W, we denote by $\mathcal{S}(p, q)$ the Schwartz space of C^∞, rapidly decreasing functions in the given chart. We shall prove that this functionnal space is in fact independent of the choice of the adapted chart.

LEMMA 2.1. *Let Φ be any polynomial diffeomorphism with polynomial inverse on $\mathbb{R}^n$. For any function u in $\mathcal{S}(\xi)$ we put:*

$$(\widetilde{\Phi}(u))(\xi) = (u \circ \phi)(\xi); \quad \xi \in \mathbb{R}^n.$$

Then $\widetilde{\Phi}$ is homeorphic from $\mathcal{S}(\xi)$ into itself with the usual Frechet topology. ∎

Proof. It is clear that $\widetilde{\phi}$ and $\widetilde{\phi}^{-1}$ preserve the set of differential operators with polynomial coefficients i.e. the set of semi-norms defining the topology of $\mathcal{S}(\xi)$. ∎

LEMMA 2.2. *Given a Jordan-Hölder basis $(e_1, \ldots, e_n)$ of g^* and W an orbit in g^*, there exists a unique indices family $j_1 < j_2 < \ldots < j_{2k}$ such that for each $\xi = \sum_{i=1}^{n} \xi_i e_i \in W$, the expression:*

$$\theta(\xi) = (\xi_{j_1}, \ldots, \xi_{j_{2k}}) \in \mathbb{R}^{2k}$$

defines a bijective map from W onto $\mathbb{R}^{2k}$.

Moreover:

1) $\xi_i = P_i(\xi_{j_1}, \ldots, \xi_{j_{2k}})$; $i = 1, 2, \ldots, n$ *where P_i is a polynomial function*

2) *The space $\mathcal{S}(W)$ of C^∞ functions u on W such that $u \circ \theta^{-1} \in \mathcal{S}(\mathbb{R}^{2k})$, is independent of the choosen Jordan-Hölder basis.* ∎

Proof. θ is defined in [11]. The independance of $\mathcal{S}(W)$ with respect to the choice of the basis results directly from lemma 2.1.

Note that the space $\mathcal{S}(W)$ was used in a similar context in [12]. ∎

PROPOSITION 2.2. *Let W be an orbit in g^*. Independently of the adapted chart C on W, the space $\mathcal{S}(W)$ is the space of C^∞ functions u on W such that $u \circ C^{-1}$ belongs to $\mathcal{S}(\mathbb{R}^{2k})$.* ∎

Proof. Using notations of Lemma 2.2, for a given Jordan-Hölden basis of g^* and a given adapted chart C, the functions

$$\xi_{j_l} \circ C^{-1} : \mathbb{R}^{2k} \to \mathbb{R} \qquad (l = 1, 2, \ldots, 2k)$$

are polynomial ana then $\theta \circ C^{-1}$ is polynomial. Using Lemma 2.1, it is sufficient, in order to prove the proposition, to show that $C \circ \theta^{-1}$ is polynomial. This can be done by induction on the dimension n of g.

Let us remark that the result is trivial for $n = 1$ or 2 since $\dim W = 0$.

Let us associate to W the linear form W on the center z of g defined by

$$W(Z) = \langle \xi, z \rangle; \quad \xi \in W, \ Z \in z.$$

If *Ker* $W \neq \{0\}$:

Let us consider $g_1 = g/\mathrm{Ker}\, W$. Then W appears as an orbit in g_1^* identified with the orthogonal space of Ker W and the induction hypothesis gives the result.

If *Ker* $W = \{0\}$

Then $z = \mathbb{R} Z$ is one dimensional and g can be written:

$$g = \mathbb{R}\, X \oplus \mathbb{R}\, Y \oplus \mathbb{R}\, Z \oplus \hat{g} = \mathbb{R}\, X \oplus g_1$$

where:

$$[X, Y] = Z \quad \text{and} \quad g_1 = \{X_1 \in g \text{ such that } [X_1, Y] = 0\}.$$

Let us identify g_1^* with $X^\perp$ and define $e_X \in g^*$ by:

$$e_X \in g_1^\perp \quad \text{and} \quad \langle e_X, X \rangle = 1$$

Then W has the form:

$$W = \bigcup_{q_k \in \mathbf{R}} (\exp - q_k X \cdot W_1 + p_k e_X)$$

where W_1 is a g_1 orbit in g_1^*. Thus we define an isomorphism:

$$W \to \mathbb{R}^2 \times W_1.$$

It is the first step of definition of the chart C, clearly: $p_k = \widetilde{X}(\xi)$, $q_k = \widetilde{Y}(\xi)$. Let $X_i \in g_1$ be a basis of g_1. In g_1^*, the coordinates of $\eta \in W_1$ are:

$$\eta_i = \langle \eta, X_i \rangle.$$

Our induction hypothesis says that for $j \leqslant (k-1)$:

$$q_j = \theta_j(\eta_{j_1}, \ldots, \eta_{j_{2(k-1)}}); \quad p_j = p_j(\eta_{j_1}, \ldots, \eta_{j_{2(k-1)}})$$

but if $[X, X_i] = 0$, we have for $\xi = \exp - q_k X \cdot \eta + p_k e_X$:

$$\xi_i = Ad \exp \widetilde{q_k X} \cdot (X_i)(\eta) = \eta_i,$$

if $(\mathrm{ad}\, X)^2 (X_i) = 0$, then:

$$\xi_i = \eta_i + q_k \langle \eta, [X, X_i] \rangle = \eta_i + q_k \widetilde{[X, X_i]}(\xi)$$

or:

$$\eta_i = \xi_i - \widetilde{Y}_k(\xi) \widetilde{[X, X_i]}(\xi).$$

Then inductively, each η_j is a polynomial function (ϵ_j) of ξ i.e.

$$q_j = Q_j(\epsilon_{j_1}(\xi) \ldots \epsilon_{j_{2k-2}}(\xi)); \quad p_j = P_j(\epsilon_{j_1}(\xi), \ldots, \epsilon_{j_{2k-2}}(\xi)).$$

But each ξ_i is a polynomial function of $\xi_{j'_1}, \ldots, \xi_{j'_{2k}}$ coordinates of ξ in a Jordan--Hölden basis of g^* thus the same is true for $p_i, q_i, i = 1, \ldots, k$. ∎

2.3. g-module structure on $(\mathscr{S}(W), *)$

Like in § 1 and § 2 - 1, let W be an orbit in g^* endowed with an adapted chart, the corresponding law * and Von Neumann algebra $\mathfrak{G}(p, q)$.

Let R be a fixed element in the space $\mathbf{C}[p, q]$ of polynomial functions in the coordinates $(p_j, q_j; j = 1, 2, \ldots, k)$. The map:

$$u \in \mathscr{S}(W) \to P^r(R, u) \quad (r \text{ fixed}, P^r \text{ defined by (6)})$$

is a differential operator with polynomial coefficients, then a continuous operator on $\mathscr{S}(W)$, vanishing if $r >$ degree of R.

Then for fixed λ, we define a continuous operator on $\mathscr{S}(W)$ by:

$$u \to R *_\lambda u \qquad (\text{resp. } u *_\lambda R)$$

where $*_\lambda$ is the Moyal product (7) on W.

LEMMA 2.3. *For R in $\mathbf{C}[p, q]$, u and v in $\mathscr{S}(W)$ and for $\lambda = i$, we have:*

$$R *_\lambda (u * v) = (R *_\lambda u) * v$$

$$u * (R *_\lambda v) = (u *_\lambda R) * v$$

$$(u * v) *_\lambda R = u * (v *_\lambda R)$$

$$\int R *_\lambda u \, dp \, dq = \int Ru \, dp \, dq$$

*where * is the Weyl Moyal product.* ∎

Proof. $\mathbf{C}[p, q]$ is a *-algebra. Moreover each $*_\lambda$-polynomial of degree r in the variables $(p_j, q_j; j = 1, 2, \ldots, k)$ is an usual polynomial of degree r. Therefore it is sufficient to prove the lemma when $R = p_j$ or $R = q_j$. Using formulas of Lemma 1.1. an easy computation leads to the identity:

$$(p_j *_\lambda F(u)) * F(v) = i\, F\left(\left(\frac{\partial u}{\partial q_j} + i\, p_j\, u\right) x_\sigma\, v\right)$$

$$= p_j *_\lambda (F(u) * F(v))$$

and analogous identity by substitution of p_j by q_j.

The last assertion is obtained by partial integration. ■

Let us denote $D(X)$ the restriction of $D_\lambda(X)$ to $\mathscr{S}(W)$ for $\lambda = i$ and $D : g \ni X \to D(X)$.

PROPOSITION 2.3. *D is a representation of g by derivations of $(\mathscr{S}(W), *)$.* ■

Proof. Lemma 2.3 proofs that $D(X)$ is a derivation of $(\mathscr{S}(W), *)$. ■

The associativity of the Moyal product gives the property of representation.

Remark 2.1. There exists on $\mathscr{S}(W)$ two other g-module structures:

$$\begin{cases} g \times \mathscr{S}(W) \longrightarrow \mathscr{S}(W) \\ \\ (X, u) \longrightarrow \dfrac{-1}{2i}\, \widetilde{X} *_\lambda u \end{cases} \quad \text{and} \quad \begin{cases} g \times \mathscr{S}(W) \longrightarrow \mathscr{S}(W) \\ \\ (X, u) \longrightarrow \dfrac{1}{2i}\, u *_\lambda \widetilde{X}. \end{cases}$$

However in both cases the elements of g are not represented by derivations on $\mathscr{S}(W)$. ■

3. REPRESENTATION OF G

3.1. Representations of G by automorphisms of $\mathfrak{G}$ (p, q)

The goal of this part is to exponentiate the above representation D of g to the corresponding connected, simply connected Lie group G.

At the one-parameter group level we have to solve the following differential equation in the group of $*$-automorphisms of $\mathscr{S}(W)$:

$$\frac{\mathrm{d}}{\mathrm{d}t} A(\operatorname{expt} X) = D(X) \circ A(\operatorname{expt} X)$$

with initial data $A(0) = \mathrm{Id}$.

Let us recall that

$$D(X)u = [\widetilde{X}, u]_{\lambda = +i} = \sum_{s=0}^{N} \frac{(-1)^s}{(2s+1)!} P^{2s+1}(\widetilde{X}, u) \quad \text{with}$$

$$(18) \qquad 2N + 1 \geqslant \mathrm{d}^0 \widetilde{X}.$$

Using the expression (17) for $\widetilde{X}$ and with the convention $p_0 = 1$ we get:

LEMMA 3.1.

$$D(X) = \sum_{s=0}^{N} \sum_{j \geq 0} \alpha_{i_1, \ldots, i_{2s+1}}(q)\, p_j \frac{\partial^{2s+1}}{\partial_{p_{i_1}} \ldots \partial_{p_{i_{2s+1}}}} +$$

$$(19) \qquad + \sum_{s=0}^{N} \sum_{j>0} \beta_{i_1, \ldots, i_{2s}, j}(q) \frac{\partial^{2s+1}}{\partial q_j \partial p_{i_1} \ldots \partial p_{i_{2s}}}$$

$$i_1, i_2, \ldots, i_{2s} > j$$

where the coefficients $\alpha_{i_1, \ldots, i_{2s+1}}$ *and* $\beta_{i_1, \ldots, i_{2s}, j}$ *are polynomial functions in the variables* q_j $(j = 1, 2, \ldots, k)$. ■

Let $\mathscr{F}_\pi$ be the (usual) partial Fourier transform with respect to the variables p_j $(j = 1, 2, \ldots, k)$. $\mathscr{F}_\pi(\mathscr{S}(p, q)) = \mathscr{S}(\pi, q)$ and let us denote $\widetilde{D}(X) = \mathscr{F}_\pi \circ \circ D(X) \circ \mathscr{F}_\pi^{-1}$.

As an immediate consequence of the expression (19) we have:

LEMMA 3.2.

$$(20) \qquad \widetilde{D}(X) = \sum_{j=1}^{k} \left(\alpha_j(\pi, q) \frac{\partial}{\partial \pi_j} + \beta_j(\pi, q) \frac{\partial}{\partial q_j} \right) + i\, a_0(\pi, q)$$

where $\alpha_j(\pi, q)$ *(resp.* $\beta_j(\pi, q)) \in \mathbb{R}[\pi_{j+1}, \ldots, \pi_k, q_{j+1}, \ldots, q_k]$

$$a_0(\pi, q) \in \mathbb{R}[\pi, q].$$ ■

Remark. The last term a_0 in $D(X)$ comes from the term obtained at $j = 0$ in (19), the only non real term in this last expression. ■

LEMMA 3.3. *With the above notations let* V_X^l *be the vector field:*

$$V_X^l(p, q) = \sum_{j=l}^{k} \left(\alpha_j(\pi, q) \frac{\partial}{\partial \pi_j} + \beta_j(\pi, q) \frac{\partial}{\partial q_j} \right) \quad (l = 1, 2, \ldots, k)$$

i) *the flow of* V_X^l *is given on each* $u \in C^\infty(\pi, q)$ *by:*

$$(21) \qquad \begin{aligned} (\operatorname{expt} V_X^l \cdot u)(\pi, q) &= u(\pi_i + A_i(t, \pi, q), q_i + B_i(t, \pi, q)) = \\ &= u(\operatorname{expt} V_X^l(\pi, q)) \end{aligned}$$

where:

$$B_k(t,\pi,q) = t\beta_k\,; \quad A_k(t,\pi q) = t\alpha_k$$

$$(22)\qquad \left.\begin{aligned} A_i(t,\pi,q) &= \int_0^t \alpha_i(\pi_j + A_j(s,\pi,q), q_j + B_j(s,\pi,q))\,\mathrm{d}s \\ B_i(t,\pi,q) &= \int_0^t \beta_i(\pi_j + A_j(s,\pi,q), q_j + B_j(s,\pi,q))\,\mathrm{d}s \end{aligned}\right\}\begin{array}{l} i = l, \dots, k-1 \\ j > i \end{array}$$

$$A_i(t,\pi,q) = B_i(t,\pi,q) = 0 \quad \text{if} \quad i < l.$$

ii) $A_i(t,\pi,q)$ *(resp.* $B_i(t,\pi,q) \in \mathbb{R}([\pi_{i+1},\dots,\pi_k, q_{i+1},\dots,q_k])$. ■

Proof. Since $\alpha_i(\pi,q)$ and $\beta_i(\pi,q)$ depend only of the π_j's and q_j's for $j > i$, we can compute the flow of V_X^l from those of V_X^i $(i > l)$.

Then from the flow of V_X^k (vector field with constants coefficients) we compute the flow of V_X^l till $l = 1$. The part ii) of the lemma is a direct consequence of the expression (22). ■

PROPOSITION 3.1. *Let X be an element of g. For a given function u in* $\mathscr{S}(\pi,q)$ *the equation:*

$$(23)\qquad \frac{\partial F}{\partial t}(t,\pi,q) = (\widetilde{D}(X)\circ F)(t,\pi,q)$$

with boundary condition $F(0,\pi,q) = u(\pi,q)$, *admits a unique solution:*

$$(24)\qquad F(t,\pi,q) = e^{i\int_0^t a_0(\exp s\, V_X^1(\pi,q))\,\mathrm{d}s}\cdot u(\exp t\, V_X^1(\pi,q))$$

$(W(t,X)u)(\pi,q) = F(t,\pi,q)$ *defines a continuous operator on* $\mathscr{S}(\pi,q)$ *which admits an unique unitary extension on* $L^2(\mathbb{R}^{2k}, \mathrm{d}p\,\mathrm{d}q)$. ■

Proof. Since $\widetilde{D}(X) = V_X^1 + i\,a_0$, the solution (24) of equation (23) is easily deductible from the flow of V_X^1 (given by (21)).

From the polynomial character of this flow and from the unitarity of $e^{i\int_0^t a_0(\exp t\, V_X^1\cdot(\pi,q))\,\mathrm{d}s}$ we deduce the stability of $\mathscr{S}(\pi,q)$ under $W(t,X)$ and the continuity of $W(t,X)u$ from $\mathbb{R}$ in $L^2(\mathbb{R}^{2k}, \mathrm{d}p\,\mathrm{d}q)$ for any C^∞ function u with compact support. Moreover $W(t,X)$ is unitary since the Jacobian of the map $(\pi,q) \to \exp t\, V_X^1(\pi,q)$ is 1. ■

$t \to W(t, X)$ defines an one parameter group of unitary operators on $L^2(\mathbb{R}^{2k}, \mathrm{d}p\,\mathrm{d}q)$. On $\mathscr{S}(\pi, q)$ the generator of this group is $\widetilde{D}(X)$. Let us denote

$$U(t, X) = \mathscr{F}_{\pi}^{-1} \circ W(t, X) \circ \mathscr{F}_{\pi}.$$

We have evidently:

PROPOSITION 3.2. *For each element X in g, there exists an unique unitary one-parameter group $U(t, X)$ on $L^2(W, \mathrm{d}p\,\mathrm{d}q)$ such that:*

i) *$\mathscr{S}(W)$ is contained in the domain of the generator of $U(t, X)$ and stable under $U(t, X)$.*

ii) *On $\mathscr{S}(W)$, $\frac{\mathrm{d}}{\mathrm{d}t} U(t, X)|_{t=0} = D(X)$.* ■

PROPOSITION 3.3. i) *There exists an unique unitary representation U of G in $L^2(W, \mathrm{d}p\,\mathrm{d}q)$ such that:*

$$\frac{\mathrm{d}}{\mathrm{d}t} U(\exp t X) = D(X) \quad on \quad \mathscr{S}(W); \quad X \in g.$$

ii) *Let $\alpha(g) : \mathscr{S}(W) \to \mathscr{S}(W)$ be the restriction of $U(g)$ to $\mathscr{S}(W)$ $(g \in G)$. Then the map $g \in G \to \alpha(g)$ is a representation of G in the group of (continuous) $*$-automorphisms of $\mathscr{S}(W)$.* ■

Proof. i) Let $g_1 = \exp X$ and g_2 be elements of G. For u in $\mathscr{S}(W)$:

$$\frac{\mathrm{d}}{\mathrm{d}t} U(g_2)\, U(\exp t X)\, U(g_2^{-1})u\,|_{t=0} = U(g_2) \circ D(X) \circ U(g_2^{-1})u.$$

Now by a Taylor expansion in $s = 0$, we verify that for any Z in g we have:

$$U(\exp sZ)\, D(X)\, U(\exp -sZ)u = D(\mathrm{Ad}\,(\exp sZ))u$$

and then,

$$U(g_2) \circ D(X) \circ U(g_2^{-1})u = D(\mathrm{Ad} g_2(X))u.$$

Therefore the generator of the one-parameter group $U(g_2) \circ U(\exp tX) \circ U(g_2^{-1})$ coincides on $\mathscr{S}(W)$ with $D(\mathrm{Ad}\, g_2(X))$ and then for any t in $\mathbb{R}$:

$$U(g_2) \circ U(\exp tX) \circ U(g_2^{-1}) = U(\exp t\, \mathrm{Ad}\, g_2(x)). \tag{25}$$

In order to prove that U is a representation of G, we consider a sequence g_j of ideals in g such that $\dim g_j = j$. Let G_j be the connected subgroup of G with Lie algebra g_j.

The restriction of U to G_1 is a representation. Let us suppose that the restriction of U to G_{j-1} is also a representation. Then each $g \in G_j$ can be written $g = g_1 \cdot \exp tX$ where $g_1 \in G_{j-1}$ and $x \notin G_{j-1}$.

We have (25):

$$U(\exp tX)\, U(\exp Y)\, U(\exp - tX) = U(\exp(\mathrm{Ad}\exp tX(Y))); \; Y \in g_{j-1}.$$

From the induction hypothesis we obtain

$$V(g\,g') = V(g)\, V(g'); \quad g, g' \in G_j$$

where $V(g) = U(g_1)\, U(\exp tX)$.

At least if $Z \in g_j$, $Z = Y + \lambda X$, we have: $\exp tZ = \exp Y(t) \exp \lambda(t) X$ where $t \to Y(t)$ (reps. $\lambda(t)$) is a polynomial function of $\mathbb{R}$ in g_{j-1} (resp. $\mathbb{R}$) such that:

$$Y(0) = 0, \; \frac{\partial Y}{\partial t}\Big|_{t=0} = Y \qquad \left(\text{resp. } \lambda(0) = 0, \; \frac{\partial \lambda}{\partial t}\Big|_{t=0} = \lambda\right).$$

On $\mathscr{S}(W)$ the generator of the one-parameter group $V(\exp tZ)$ is

$$\mathrm{d}\, V(Z) = \frac{\mathrm{d}}{\mathrm{d}t}\, (U(\exp Y(t))\, U(\exp \lambda(t) X))\,|_{t=0} = \mathrm{d}\, U(Y + \lambda X).$$

Then V and U coincide on G_j.

ii) Let u and v be elements of $\mathscr{S}(W)$. We easily verify that:

$$\begin{aligned}
&\frac{\partial}{\partial t}\, U(\exp - tX)\, [U(\exp tX) u * U(\exp tX) v] = \\
&- U(\exp - tX)\, D(X)\, [U(\exp tX) u * U(\exp tX) v] \\
&+ U(\exp - tX)\, [(D(X)\, U(\exp tX) u) * U(\exp tX) v] \\
&+ U(\exp - tX)\, [U(\exp tX) u * (D(X)\, U(\exp tX) v] = 0.
\end{aligned}$$

It results from proposition 3.1 that $\alpha(\exp tX)$ is a continuous $*$ automorphism of $\mathscr{S}(W)$. ■

THEOREM 3.1. *There exists a unique representation τ_g of G in the group of automorphisms of $\mathfrak{S}(p, q)$ such that:*

(26) $$\frac{\mathrm{d}}{\mathrm{d}t}\, \tau_{\exp tX}(l_u)\,|_{t=0} = [D(X), l_u] = l_{D(X)u}; \quad u \in \mathscr{S}(W), \; X \in g.$$

This representation is defined by:

(27) $$\tau_g(l_u) = U(g) \circ l_u \circ U(g^{-1}) = l_{U(g)u}; \quad u \in \mathscr{S}(W), \; g \in G.$$ ■

Proof. As a matter of fact α_g being continuous, the map $l_u \to l_{U(g)u}$ can be extended to $\mathfrak{G}(p,q)$, strong closure of $l_{\mathscr{S}}$. This extension defines a representation τ_g satisfying (27).

Then for v in $\mathscr{S}(W)$.

$$\tau_g(l_u)(v) = (U(g)u) * v = U(g)(u * U(g^{-1})(v)) = U(g) \circ l_u \circ U(g^{-1})(v).$$

By differentiation, this expression gives (26) and the unicity of τ comes from the density of $l_{\mathscr{S}}$ in $\mathfrak{G}(p,q)$. ∎

Remark 3.1. The construction of $\mathfrak{G}(p,q)$ and U depends of the choosen adapted chart. However we shall see in §3.3 that the passage from an adapted chart to an other one induces an unitary equivalence between these objects which are then essentially unique. ∎

Remark 3.2. A similar construction can be performed with the Lie algebra representations defined in remark 2.1. Indeed the operators on $\mathscr{S}(W)$ defined by:

$$u \in \mathscr{S}(W) \to -\frac{1}{2i}\tilde{X} *_\lambda u \quad \left(\text{resp.}\ u \in \mathscr{S}(W) \to \frac{1}{2i} u *_\lambda \tilde{X}\right)$$

are differential operators of the same type as $D(X)$. Then we have for these representations similar properties as those of lemma 3.2 and propositions 3.1, 3.2, 3.3.i) for D. $\mathscr{S}(W)$ is stable under the corresponding representations of the group G in $L^2(\mathbb{R}^{2k}, dp\,dq)$. However these representations do not act on $\mathfrak{G}(p,q)$ by automorphisms. ∎

3.2. Gelfand-Naimark-Segal (G.N.S.) representation of $\mathfrak{G}$ (p, q)

In this part $*_\lambda$ (resp. $*$) denotes the Moyal product for $\lambda = i$ (resp. the Weyl Moyal product) on W in a fixed adapted chart (p,q). $dp\,dq$ being the Liouville measure on W is independant of this chart, we shall denote by L^2 the space $L^2(W, dp\,dq)$.

LEMMA 3.4. 1) *The real solutions of the equation:*

$$(28) \qquad (p_j - iq_j) * u = 0 \qquad (j = 1, \ldots, k)$$

are $u_j(p,q) = f(p,q)\, e^{-\frac{p_j^2+q_j^2}{2}}$ *where* $f(p,q) \in C^\infty(W)$ *and satisfies to* $\dfrac{\partial f}{\partial p_j} = \dfrac{\partial f}{\partial q_j} = 0$.

2) *Let us put* $\Omega(p,q) = 2^k e^{-\frac{p^2+q^2}{2}} \in \mathscr{S}(W)$. *Then:*

(29) $$\| \Omega \|_{L^2} = (2\pi)^k$$

(30) $$(p_j + iq_j) *_\lambda \Omega = 2(p_j + iq_j)\Omega$$

(31) $$\Omega * u * \Omega = (u \mid \Omega)_{L^2}(4\pi)^{-k}\Omega$$

(32) $$\Omega * \Omega = \Omega.$$ ∎

Proof. 1) (28) can be written:

$$\left(p_j u + \frac{\partial u}{\partial p_j}\right) - i\left(q_j u + \frac{\partial u}{\partial q_j}\right) = 0$$ and the solutions follow easily.

2) (29) and (30) are obtained from a direct computation. Note that the conjugate expressions of (28) and (29) give:

$$\Omega *_\lambda (p_j - iq_j) = 2(p_j - iq_j)\Omega$$

$$\Omega *_\lambda (p_j + iq_j) = 0.$$

On the other hand it is easy to verify that:

$$F\Omega = \Omega.$$

Then (31) is a consequence of $\Omega_{x_\sigma} F(u)_{x_\sigma} \Omega = (u \mid \Omega)_{L^2}(4\pi)^{-k}\Omega$ ([7]) and it is easy to verify directly (32). ∎

LEMMA 3.5. *For every multiindice $l = (l_1, \ldots, l_k) \in \mathbb{N}^k$ with length $|l| = \sum_{j=1}^{k} l_j$ we define:*

(33) $$\Omega_l(p, q) = \frac{1}{2^{|l|}(l!)^{1/2}(4\pi)^{k/2}} \mathop{*_\lambda}_{j=1}^{k} (p_j + iq_j)^{*_\lambda l_j} *_\lambda \Omega.$$

1) $\Omega_l \in \mathscr{S}(W)$ *and:*

(34) $$(p_{j_0} + iq_{j_0}) *_\lambda \Omega_l = 2(l_{j_0} + 1)^{1/2}\Omega_{l+1(j_0)}$$

(35) $$(p_{j_0} - iq_{j_0}) *_\lambda \Omega_l = 2\, l_{j_0}^{1/2}\Omega_{l-1(j_0)} \quad \text{if} \quad l \neq 0$$
$$= 0 \qquad \text{if} \quad l = 0$$

$$(l + \epsilon 1(j_0)) = (l_1, l_2, \ldots, l_{j_0-1}, l_{j_0} + \epsilon, l_{j_0+1}, \ldots, l_k)$$

(36) $$(p_{j_0}^2 + q_{j_0}^2) *_\lambda \Omega_l = (2 + 4\, l_{j_0})\, \Omega_l$$

(37) $$(p^2 + q^2) *_\lambda \Omega_l = (2k + 4|l|)\, \Omega_l.$$

2) *The set $\{\Omega_l, l \in \mathbb{N}^k\}$ is orthonormal in $L^2(W, \mathrm{d}p\, \mathrm{d}q)$ and is maximal in*

$$K = \{ u \in \mathscr{S}(W) / u * \Omega = u \}.$$

3) *The set* $\{ \Omega_l * \bar{\Omega}_m, (l, m) \in \mathbb{N}^k \times \mathbb{N}^k \}$ *is an orthonormal complete set of* $L^2(W, \mathrm{d}p\, \mathrm{d}q)$. ■

Proof. 1) From (33), we see that $\Omega_l \in \mathscr{S}(W)$. One the other hand, we have:

$$p_{j_0}^2 + q_{j_0}^2 = (p_{j_0} + iq_{j_0}) *_\lambda (p_{j_0} - iq_{j_0}) + 2 = (p_{j_0} - iq_{j_0}) *_\lambda (p_{j_0} + iq_{j_0}) - 2.$$

This gives immediatly:

$$(38) \qquad (p_{j_0}^2 + q_{j_0}^2) * \Omega = 2\Omega \quad \text{and} \quad (p^2 + q^2) * \Omega = 2k\Omega.$$

On the other hand the relation:

$$(39) \qquad (p_{j_0}^2 + q_{j_0}^2) *_\lambda (p_j + iq_j)^{\overset{l}{*\lambda}} = (p_j + iq_j)^{\overset{l}{*\lambda}} *_\lambda (p_{j_0}^2 + q_{j_0}^2) + 4l\delta_{jj_0}(p_j + iq_j)^{\overset{l}{*\lambda}};$$

$l \in \mathbb{N}^k$, $j, j_0 = 1, \dots, k$, holds.

From (38) and (39) we get:

$$(40) \qquad (p_{j_0}^2 + q_{j_0}^2) \overset{k}{\underset{j=1}{*\lambda}} (p_j + iq_j)^{\overset{l_j}{*\lambda}} * \Omega = (2k + 4l_{j_0}) \overset{k}{\underset{j=1}{*\lambda}} (p_j + iq_j)^{\overset{l_j}{*}} * \Omega$$

$$(41) \qquad (p^2 + q^2) \overset{k}{\underset{j=1}{*\lambda}} (p_j + iq_j)^{\overset{l_j}{*\lambda}} = \overset{k}{\underset{j=1}{*\lambda}} (p_j + iq_j)^{\overset{l_j}{*\lambda}} *_\lambda (p^2 + q^2) + 4|l| \overset{k}{\underset{j=1}{*\lambda}} (p_j + iq_j)^{\overset{l_j}{*\lambda}}.$$

From (41) we deduce (36) and (37).

(34) is immediate and (35) a consequence of a calcul using (36).

2) The orthogonality of the Ω_l's comes from the property

$$((p_{j_0}^2 + q_{j_0}^2) *_\lambda \Omega_l \mid \Omega_m)_{L^2} = (\Omega_l \mid (p_{j_0}^2 + q_{j_0}^2) *_\lambda \Omega_m) \quad \text{and,}$$

$$(36) \qquad \| \Omega_l \|^2 = \frac{1}{4 l_{j_0}} \| (p_{j_0} + iq_{j_0}) *_\lambda \Omega_{l-1(j_0)} \|^2$$

$$= \| \Omega_{l-1(j_0)} \|^2 ; j_0 = 1, 2, \dots, k \text{ and then } \| \Omega_l \|^2 = \| \Omega_{00\dots 0} \|^2 = 1.$$

We put:

$$K = \{ u \in \mathscr{S}(W) / u * \Omega = u \} = \{ u * \Omega / u \in \mathscr{S}(W) \}.$$

Let us consider the Hermite function in $L^2(W, \mathrm{d}p\, \mathrm{d}q) : h_l = \prod_{j=1}^{2k} h_{l_j}^j$ where:

$$h_{l_j}^j(X_j) = \frac{(-1)^{l_j}}{(l_j! 2^{l_j} \sqrt{\pi})^{1/2}} \cdot e^{\frac{X_j^2}{2}} \cdot \frac{\mathrm{d}^{l_j}}{\mathrm{d} X_j^{l_j}} e^{-X_j^2} \quad (j = 1, \dots, 2k, l_j\ \mathbb{N}).$$

Where $X_j = p_j$ if $1 \leqslant j \leqslant k$, $X_j = q_j$ if $k+1 \leqslant j \leqslant 2k$.

If $u = \sum_{l \in \mathbb{N}^{2k}} C_l \prod_{j=1}^{2k} h_{l_j}^j$ is the expansion of an element $u \in K \subset L^2(W, \mathrm{d}p\, \mathrm{d}q)$ on these functions, $u_n = \sum_{|l| \leqslant n} C_l \prod_{j=1}^{2k} h_{l_j}^j$ converges in $L^2(W, \mathrm{d}p\, \mathrm{d}q)$ to u. (the convergence is in fact uniform on W).

Since $u \in K$, the relations $u *_\lambda (p_j + iq_j) = 0$ $(j = 1, 2, \ldots, k)$ i.e.:

$$\left(\left(p_j + \frac{\partial}{\partial p_j}\right) + i\left(q_j + \frac{\partial}{\partial q_j}\right)\right) u = 0 \quad (j = 1, 2, \ldots, k) \tag{42}$$

hold.

But if $h_l = \prod_{j=1}^{2k} h_{l_j}^j$ $(l \in \mathbb{N}^{2k})$: $\left(X_j + \dfrac{\partial}{\partial X_j}\right) h_l = (2l_j)\, h_{l-1(j)}$; $j = 1, 2, \ldots, 2k$.

Then u_n satisfies (42) for each n.

Writting u_n on the form:

$$u_n = P_n(Z, \bar{Z})\, e^{-\frac{1}{2}Z\bar{Z}} \quad (Z_j = p_j + iq_j \text{ and } P_n \text{ a polynomial function}),$$

we have $\dfrac{\partial P_n}{\partial \bar{Z}_j} = 0$ $(j = 1, 2, \ldots, k)$ i.e. $P_n(Z, \bar{Z}) = Q_n(Z)$.

Since $\Omega_l = \dfrac{1}{(l!)^{1/2}(\pi)^{k/2}} R_l(Z)\, e^{-\frac{Z\bar{Z}}{2}}$ with R_l polynomial function of higher term Z^l, u_n is necessarily of the form:

$$u_n = \sum_{|l| \leqslant n} \alpha_l\, \Omega_l \quad \text{with} \quad \Sigma\, |\alpha_l|^2 < \|u\|^2.$$

Thus the series $\sum_l \alpha_l\, \Omega_l$ converges in L^2 and its limit is u since:

$$\|u - \sum_{|l| \leqslant n} \alpha_l\, \Omega_l\|^2 = \|u - u_n\|^2 \to 0.$$

The set $\{\Omega_l\}$ is total in K.

3) Using formula (35), we obtain:

$$(\Omega_m * \bar{\Omega}_l \mid \Omega_{m'} * \bar{\Omega}_{l'})_{L^2} = \left(\frac{l'_{j_0}}{l_{j_0}}\right)^{1/2} (\Omega_m * \bar{\Omega}_{l-1(j_0)} \mid \Omega_{m'} * \bar{\Omega}_{l'-1(j_0)})_{L^2}$$

from which we easily deduce the orthogonality relations and the normalization of the $\Omega_l * \bar{\Omega}_m$'s.

It can be easily obtained from (34), (35) that

$$(p_{j_0} \mp iq_{j_0})\Omega_l * \overline{\Omega}_m = 2(l_{j_0} + \epsilon)^{1/2}\Omega_{l \mp 1(j_0)} * \overline{\Omega}_m + i\{(p_{j_0} \mp q_{j_0})\Omega_l * \overline{\Omega}_m\}$$

is a linear combination of elements $\Omega_{l,m}(l, m \in \mathbb{N}^k)$. The same fact holds for all functions of the form $R(p,q) \cdot e^{-\frac{p^2+q^2}{2}}$ ($R(p,q)$ polynomial function) which are dense in $L^2(W, \mathrm{d}p\, \mathrm{d}q)$. This proves that $\Omega_l * \overline{\Omega}_m$ is a orthonormal basis of $L^2(W, \mathrm{d}p\, \mathrm{d}q)$. ∎

(32) means that Ω is a projector in $\mathfrak{G}(p,q)$ and (31) allows us to define an associate state on $\mathfrak{G}(p,q)$ defined by:

(43) $$\omega(T) = (4\pi)^{-k}(\Omega \mid T\Omega)\ (T \in \mathfrak{G}(p,q)).$$

More precisely we have following proposition derived essentially from formulas (29) to (32) and the strong density of $l_{\mathscr{S}}$ in $\mathfrak{G}(p,q)$:

PROPOSITION 3.4. 1) *The formula* $\omega(T) = (4\pi)^{-k}(\Omega \mid T\Omega)$ *for each* T *in* $\mathfrak{G}(p,q)$ *defines a state* ω *on* $\mathfrak{G}(p,q)$.

2) *If* $u \in \mathscr{S}(\mathbb{R}^{2k})$,

$$\omega(l_u) = (4\pi)^{-k}\int e^{-\frac{1}{2}(p^2+q^2)}\overline{u}(p,q)\, \mathrm{d}p\, \mathrm{d}q.$$

3) *If* $T \in \mathfrak{G}(p,q)$, $\omega(T^*T) = 0$ *if and only if* $T\Omega = 0$. ∎

In order to construct the G.N.S. representation associated to ω, let us consider the ideal $\mathscr{T}_\omega$ in $\mathfrak{G}(p,q)$:

$$\mathscr{T}_\omega = \{T \in \mathfrak{G}(p,q)/T\Omega = 0\}.$$

Then the space $\mathfrak{G} / \mathscr{T}_\omega$ is canonically endowed with a scalar product defined by:

(43) $$(\overline{T} \mid \overline{S})_\omega = \omega(T^*S) \qquad (T, S \in \mathfrak{G}(p,q)),$$

where $T \to \overline{T} \in \mathfrak{G} / \mathscr{T}_\omega$ is the canonical projection of $T \in \mathfrak{G}(p,q)$. Let $\mathscr{H}_\omega$ be the completion of $\mathfrak{G} / \mathscr{T}_\omega$ with respect to $(\ \mid\ )_\omega$.

Let us define the map Φ from K into $\mathscr{H}_\omega$ by:

(44) $$\phi(u) = (4\pi)^{k/2}\overline{l}_u; \quad u \in K.$$

$$\|\overline{l}_u\|_\omega^2 = \omega(l_{\overline{u} * u}) = (4\pi)^{-k} \cdot \|u\|_{L^2}^2 \quad \text{i.e.} \quad \|\phi(u)\|_\omega = \|u\|_{L^2}.$$

On the other hand, if $\overline{T} \in \mathscr{H}_\omega$ $\forall \epsilon > 0, \exists u \in \mathscr{S}(W)/ \|\overline{T} - \overline{l}_u\|_\omega = \|(T - l_u)\Omega\|_{L^2} < \epsilon$ (strong density of $l_{\mathscr{S}}$ in $\mathfrak{G}(p,q)$).

Then ϕ^{-1} is an isometric map between the space $\mathscr{H}_\omega$ of the G.N.S. representation and the closure $\overline{K}$ of K in $L^2(W, \mathrm{d}p\, \mathrm{d}q)$.

However instead of use this realization of $\mathcal{H}_\omega$, we give an other unitary equivalent form of $\mathcal{H}_\omega$, identifying $\overline{K}$ (and then $\mathcal{H}_\omega$) with $L^2(\mathbb{R}^k, \mathrm{d}x)$.

LEMMA 3.6. 1) *We define an unitary map* ψ *from* $\overline{K}$ *on* $L^2(\mathbb{R}^k, \mathrm{d}x)$ *by*

$$\psi(\Omega_l) = k_l(x) = \frac{(-i)^{|l|}}{(l!2^{|l|}\sqrt{\pi})^{1/2}}\, e^{\frac{x^2}{2}} \frac{\mathrm{d}^{|l|}}{\mathrm{d}x^l} e^{-x^2} \quad (l \in \mathbb{N}^k,\ x \in \mathbb{R}^k).$$

2) *We consider on* $L^2(\mathbb{R}^k, \mathrm{d}x)$ *the operators* $a_j^{\mp}$ $(j = 1, 2, \ldots, k)$ *defined on* $\mathcal{S}(\mathbb{R}^k)$ *by*

$$a_j^+ = x_j - \frac{\partial}{\partial x_j} \qquad a_j^- = x_j + \frac{\partial}{\partial x_j}\ .$$

Then, for $j = 1, 2, \ldots, k$:

$$\psi((p_j + iq_j) *_\lambda \Omega_l) = i\sqrt{2}\, a_j^+(k_l) = 2(l_j + 1)^{1/2} k_{l+1(j)} \tag{45}$$

$$\psi(p_j - iq_j) *_\lambda \Omega_l) = -i\sqrt{2}\, a_j^-(k_l) = 2l_j^{1/2} k_{l-1(j)} \quad \text{if} \quad l_j > 0 \tag{46}$$

$$= 0 \quad \text{if} \quad l_j = 0.$$ ■

Proof. 1) holds since the k_l's are (up to a Fourier transform) the usual Hermite functions in $L^2(\mathbb{R}^k, \mathrm{d}x)$,

2) is a consequence of (34) and (35) and of well known relations on the Hermite functions (see for instance [11]). ■

Now with the map $\psi \circ \phi^{-1}$, we have identified $L^2(\mathbb{R}^k, \mathrm{d}x)$ with $\mathcal{H}_\omega$. Using the definition of the GNS representation π_ω: $\pi_\omega(T)(\overline{S}) = \overline{TS}$; $T, S \in \mathfrak{G}(p, q)$, we obtain the following realization of the so-called Schrödinger representation of $\mathfrak{G}\ (p, q)$:

THEOREM 3.2. 1) *The G.N.S. representation* π_ω *of* $\mathfrak{G}\ (p, q)$ *associated to the state* ω *is given on the Hilbert space*

$$\mathcal{H}_\omega = L^2(\mathbb{R}^k, \mathrm{d}x)$$

by:

$$\pi_\omega(T) = \psi \circ T \circ \psi^{-1}(f); \quad T \in \mathfrak{G}(p, q),\ f \in \mathcal{H}_\omega,$$

with cyclic vector Ω_ω, $\Omega_\omega(x) = k_0(x) = \dfrac{1}{(\sqrt{\pi})^{1/2}}\, e^{-\frac{x^2}{2}}$.

2) *In particular, if* $u \in \mathcal{S}(W)$

(47)
$$(\pi_\omega(l_u))(k_l) = \psi(u * \Omega_l); \qquad l \in \mathbb{N}^k.$$
$$\overline{l}_u = \pi_\omega(l_u) k_0 = \psi(u * \Omega_0).$$
$$\overline{l}_{\Omega_l} = (4\pi)^{-k/2} k_l.$$
■

We denote by θ the map defined from $\mathcal{S}(W)$ into $L^2(\mathbb{R}^k, \mathrm{d}x)$ by

$$\theta(u) = \overline{l}_u; \quad u \in \mathcal{S}(W).$$

Using (47), (45) and (46) can be written:

(45′)
$$\theta((p_j + iq_j) *_\lambda \Omega_l) = 2(l_{j+1})^{1/2} \theta(\Omega_{l+1(j)})$$

(46′)
$$\theta((p_j - iq_j) *_\lambda \Omega_l) = 2 l_j^{1/2} \theta(\Omega_{l-1(j)}) \quad \text{if} \quad l_j > 0$$
$$= 0 \qquad \text{if} \quad l_j = 0.$$

Linear combinations of (45′) and (46′) give:

LEMMA 3.7. *For each finite linear combination* u *of the* Ω_l*'s, we have*

(48)
$$\left.\begin{aligned} &\theta(p_j *_\lambda u) = -i\sqrt{2}\,\frac{\partial}{\partial x_j}\,\theta(u) \\ &\theta(q_j *_\lambda u) = 2\,x_j\,\theta(u) \\ &\theta(u *_\lambda p_j) = \theta(u *_\lambda q_j) = 0 \end{aligned}\right\} j = 1, \ldots, k.$$
■

Remark 3.3. The last line in (48) is obtained with the equality $(u *_\lambda p_i) = = \theta(u *_\lambda p_i * \Omega)$. (48) can be easily extended by continuity to any u in $\mathcal{S}(W)$. ■

Remark 3.4. π_ω is clearly faithful since:

$$\pi_\omega(T) = 0$$

implies:

$$\psi(Tu) = 0; \quad u \in K.$$

In particular:

$$T\Omega_l = 0; \quad l \in \mathbb{N}^k.$$

Now let $r_{\overline{\Omega}_m}$ be the bounded operator on $L^2(W, \mathrm{d}p\,\mathrm{d}q)$ such that

$$r_{\overline{\Omega}_m}(u) = u * \overline{\Omega}_m\,; \quad u \in \mathscr{S}(w).$$

Of course

$$l_v \circ r_{\overline{\Omega}_m} = r_{\overline{\Omega}_m} \circ l_v\,; \quad v \in \mathscr{S}(w).$$

Then T belonging to $l''_{\mathscr{S}}$ satisfies;

$$T \circ r_{\overline{\Omega}_m} = r_{\overline{\Omega}_m} \circ T$$

i.e.:

$$T(\Omega_l * \overline{\Omega}_m) = r_{\overline{\Omega}_m}(T\Omega_l) = 0 \qquad (l, m \in \mathbb{N}^k) \quad \text{i.e.} \quad T = 0.$$

π_ω is irreducible: if $A \in \mathscr{L}(\mathscr{H}_\omega)$ commutes with each $\pi_\omega(T)$, $T \in \mathfrak{G}(p,q)$, it commutes with $\pi_\omega(l_{\Omega_l} * \overline{\Omega}_m)$ but:

$$\pi_\omega(l_{\Omega_l * \overline{\Omega}_m})\, k_n = \delta_{mn}\, k_l; \quad m, n, l \in \mathbb{N}^l.$$

Thus,

$$Ak_l = A\, \pi_\omega(l_{\Omega_l * \overline{\Omega}_l})\, k_l = (l_{\Omega_l * \overline{\Omega}_l})\, Ak_l; \quad l \in \mathbb{N}^k,$$

which implies:

$$Ak_l = \alpha_l\, k_l; \quad l \in \mathbb{N}^k\,;$$

and:

$$\pi_\omega(l_{\Omega_m * \overline{\Omega}_l})\, Ak_l = \alpha_l\, k_m = \alpha_m\, k_m.$$

Thus

$$A = \alpha\, \mathrm{Id}.$$

3.3. Unitary representations of G

We first recall some results in the usual theory of orbits by Kirillov ([3]).

If $X \in g$, we knew that in an adapted chart (p, q) on W:

$$\widetilde{X}(p,q) = \sum_{j=0}^{k} \alpha_j(q)\, p_j \quad \text{(with convention } p_0 = 1) \; (17).$$

We consider the vector field X^- on $\mathbb{R}^k$ defined by:

$$X^- = \frac{1}{\sqrt{2}} \sum_{j=1}^{k} \alpha_j\,(\sqrt{2}x)\, \frac{\partial}{\partial x_j}\,.$$

Then, the theory of orbits assigns to W the irreducible unitary representation R of

G defined on $L^2(\mathbb{R}^k, \mathrm{d}x)$ by:

$$(49) \qquad (R(\exp X)f)(x) = e^{i/2 \int_0^1 \alpha_0(\sqrt{2}\exp - sX^- \cdot x)\,\mathrm{d}s} f(\exp - X^- \cdot x); \quad f \in L^2(\mathbb{R}^k, \mathrm{d}x), X \in g$$

$$(49') \qquad = e^{i\gamma(g,x)} f(x \cdot g) \text{ with obvious notations.}$$

In fact the peculiar form (49) of R can be established by induction on the dimension of G. Let us brievly describe this construction.

As in proposition 2.2 we distinguish two cases according to the kernel of the form W on the center z of g ($W(Z) = \langle \xi, Z \rangle$; $Z \in z$, $\xi \in W$). We refer to proposition 2.2 for notations.

If Ker $W \neq 0$, then R can be considered as the corresponding representation of G_1 with Lie algebra $g/\mathrm{Ker}\ W$.

If Ker $W = 0$, then R is induced by a unitary representation R_1 of G_1 ($g = \mathbb{R}X + g_1$) acting on $L^2(\mathbb{R}^{k-1}, \mathrm{d}x_1 \ldots \mathrm{d}x_{k-1})$ ([11]).

With identitication of square integrable functions from $\mathbb{R} = G/G_1$ into $L^2(\mathbb{R}^{k-1}, \mathrm{d}x)$ and functions in $L^2(\mathbb{R}^k)$ we can write:

$$(R(\exp X_1 \exp tX)f)(x_k) = R_1(\exp \mathrm{Ad} \exp - x_k X(X_1))\, f(x_k - t)$$
$$= f([\exp - \mathrm{Ad} \exp - x_k X(X_1)]^-(x_1, \ldots, x_{k-1}), x_k - t)$$

But a direct computation shows that:

$$[(\exp - tX^- \circ \exp - X_1^-)x_j = [[\exp - \mathrm{Ad} \exp - x_k X(X_1)]^-(x_1, \ldots, x_{k-1})]_j \quad \text{for } j < k$$
$$= x_k - t \quad \text{if} \quad j = k.$$

The multiplier is thus:

$$\exp\left[i/2 \int_0^1 \alpha_{ox_k}(\sqrt{2}[\exp - s(\mathrm{Ad} \exp - x_k X(X_1))]^-(x_1, \ldots, x_{k-1}))\,\mathrm{d}s\right]$$

where $\alpha_{ox_k}(q_1, \ldots, q_{k-1})$ is the term with degree 0 in p in Ad $\exp - x_k X(X_1)$ on the orbit W_1 of G_1 throught 0. We deduce from proposition 2.2 that $\alpha_{ox_k}(q_1, \ldots, q_k) = \alpha_o(x_1, \ldots, x_k)$ and the multiplier is then:

$$\exp\left(i/2 \int_0^1 \alpha_0(\sqrt{2} \exp - sX^- \cdot x)\,\mathrm{d}s\right).$$

In fact (49) differs from the usual presentation of Kirillov's theory by the fact that, for normalization reasons, we use as inducing character for R half of the usual one.

We shall now prove that in the GNS construction of $\mathfrak{G}(p,q)$ the representation of G constructed in theorem 3.1 is unitarily implemented by R.

THEOREM 3.3. *Using notations of theorem 3.1*

$$\pi_\omega(\tau_g T) = R(g)\,\pi_\omega(T)\,R(g^{-1});\quad T\in \mathfrak{G}(p,q),\quad g\in G. \qquad ■$$

Proof. Lemma 3.7. implies on each finite linear combination u of the Ω_l's that if $X\in g$, $\widetilde{X}(p,q) = \sum_{j=1}^{k} \alpha_j(q)p_j + \alpha_0(q)$ and $\alpha_0(q) = \sum_l \alpha_{0,l}\, q^l$,

$$\theta(D(X)u) = \left(\frac{1}{\sqrt{2}}\sum_{j=1}^{k} \alpha_j(\sqrt{2}x)\,\frac{\partial}{\partial x_j} + \frac{i}{2}\,\alpha_0(\sqrt{2}x)\right)\theta(u) -$$

(50) $$-\frac{i}{2\sqrt{2}}\sum_m \alpha_{0,2m}\,\frac{(2m)!}{m!\,2^{|m|}}\,\theta(u) = T(X)\theta(u) \text{ where}$$

$T(X)$ is defined on $\mathscr{S}(\mathbb{R}^k)$ by:

$$T(x) = \left(\frac{1}{\sqrt{2}}\sum_{j=1}^{k} \alpha_j(\sqrt{2}x)\,\frac{\partial}{\partial x_j} + \frac{i}{2}\,\alpha_0(\sqrt{2}x)\right) - \frac{i}{2\sqrt{2}}\sum_m \alpha_{0,2m}\,\frac{(2m)!}{m!\,2^{|m|}} =$$

$$= X^- + \frac{i}{2}\,\alpha_0(\sqrt{2}x) + i\,C_0 \quad \text{where} \quad C_0\in\mathbb{R}.$$

If v is a finite linear combination of the $\Omega_l * \overline{\Omega}_m$'s then,

$$\frac{d}{dt}\,\pi_\omega(\tau_{\exp tX} l_v)(\theta(u))\big|_{t=0} = \pi_\omega(l_{D(X)v})(\theta(u)) \qquad \text{(th.3.1)}$$

$$= \pi_\omega(l_{(D(X)v)*u})\,k_0 \qquad \text{(th.3.2)}$$

$$= \theta((D(X)v) * u)$$

$$= \theta(D(X)(v * u)) - \theta(u * D(X)v)$$

$$= T(X)\circ\pi_\omega(l_v)(\theta(u)) - \pi_\omega(l_v)\circ T(X)(\theta(u)) \quad \text{using (50)}$$

$$= [T(X), \pi_\omega(l_v)]\,\theta(u).$$

Then we have:

$$(51) \qquad \pi_\omega(\tau_{\exp X}\, l_v) = e^{T(X)}\pi_\omega(l_v)\, e^{-T(X)} \text{ on } L^2(\mathbb{R}^k, \mathrm{d}x),$$

the exponentiation of $T(X)$ being very similar to these of $D(X)$ in proposition 3.1.

(51) is then extended to:

$$\pi_\omega(\tau_{\exp X}\, S) = e^{T(X)}\pi_\omega(S)\, e^{-T(X)}$$

for S in $l_{\mathcal{S}}$ and then in $\mathfrak{G}\ (p,q)$.

But,

$$(R(\exp - X)\, e^{T(X)}\, f)(x) =$$

$$= e^{\gamma(\exp - X, x)} e^{i/2 \int_0^1 \alpha_0 (\sqrt{2} \exp - sX \cdot \exp - X \cdot x)\, \mathrm{d}x} e^{iC_0} f(x \cdot \exp X \cdot \exp - X) =$$

$$= e^{\gamma(\exp - X, x)} e^{-\gamma(\exp - X, x)} e^{iC_0} f(x), \qquad (f \in L^2(W, \mathrm{d}p\, \mathrm{d}q)).$$

Then for $X \in g,\ S \in \mathfrak{G}\ (p,q),\ f \in L^2(W, \mathrm{d}p\, \mathrm{d}q)$,

$$R(\exp - X) e^{T(X)}) \pi_\omega(S) e^{-T(X)} R(\exp X) f = e^{iC_0} \pi_\omega(S) e^{-iC_0} f = \pi_\omega(S) f.$$

This last relation proves our theorem. ∎

The space H.S. of Hilbert Schmidt operators on $L^2(\mathbb{R}^k, \mathrm{d}x)$ is unitarily mapped on $L^2(\mathbb{R}^{2k}, \mathrm{d}x\, \mathrm{d}y)$ by $\varphi \in L^2(\mathbb{R}^{2k}, \mathrm{d}x\, \mathrm{d}y) \to A_\varphi \in$ H.S. such that:

$$A_\varphi(f)(x) = \int_{\mathbb{R}^k} \varphi(t, x) f(t)\, \mathrm{d}t; \qquad f \in L^2(\mathbb{R}^k, \mathrm{d}x).$$

By this isometry, the unitary representation $\hat{R}$ of $G \times G$ defined on H.S. by

$$\hat{R}(g, g')(A) = R(g) \circ A \circ R(g'^{-1}); \quad (g, g') \in G \times G,\ A \in \text{H.S.},$$

is transformed in a unitary representation $\widetilde{R}$ of $G \times G$ on $L^2(\mathbb{R}^{2k}, \mathrm{d}x\, \mathrm{d}y)$ defined by:

$$A_{\widetilde{R}(g, g')\varphi} = \hat{R}(g, g')\, (A_\varphi); \qquad \varphi \in L^2(\mathbb{R}^{2k}, \mathrm{d}x\, \mathrm{d}y).$$

$$(\hat{R}(g, h) A_\varphi)(f)(x) = \int_{\mathbb{R}^k} e^{i\gamma(g, x)} \varphi(t, g^{-1} \cdot x)\, e^{i\gamma(h^{-1}, t)} f(h \cdot t)\, \mathrm{d}t =$$

$$= \int_{\mathbb{R}^k} e^{i\gamma(g, x)} \varphi(h^{-1} t, g^{-1} x)\, e^{i\gamma(h^{-1}, h^{-1} t)} f(t)\, \mathrm{d}t.$$

Then

$$(\widetilde{R}(g,h)\varphi)(x,y) = e^{i\gamma(g,y)}e^{i\gamma(h^{-1},h^{-1}x)}\varphi(h^{-1}x, g^{-1}y)$$

$$(52) \qquad (\widetilde{R}(g,h)\varphi)(x,y) = $$
$$= e^{i\gamma(g,y)}e^{-i\gamma(h,x)}\varphi(h^{-1}x, g^{-1}y); \quad \varphi \in L^2(\mathbb{R}^{2k}, \mathrm{d}x\,\mathrm{d}y); \quad g, h \in G.$$

By partial differentiation, if $X \in g$;

$$(53) \qquad \begin{aligned} (\mathrm{d}\widetilde{R}(X,0)\varphi)(x,y) &= (\mathrm{d}R_y(X)\varphi)(x,y), \\ (\mathrm{d}\widetilde{R}(0,X)\varphi)(x,y) &= (\mathrm{d}\overline{R}_x(X)\varphi)(x,y), \end{aligned}$$

on the space $L^2(\mathbb{R}^{2k}, \mathrm{d}x\,\mathrm{d}y)^\infty$ of differentiable vectors for $\widetilde{R}$, where $\mathrm{d}R_y(X)$ means the action of $\mathrm{d}R(X)$ on the variable y and $\mathrm{d}R_x(X)$ the action of the complex conjugate of $\mathrm{d}R(X)$ on x.

PROPOSITION 3.5. *The space* $L^2(\mathbb{R}^{2k}, \mathrm{d}x\,\mathrm{d}y)^\infty$ *(resp.* H.S.$^\infty$*) of* C^∞ *vectors with respect to the unitary representation* $\widetilde{R}$ *(resp.* $\hat{R}$*) of* $G \times G$ *are:*

$$L^2(\mathbb{R}^{2k}, \mathrm{d}x\,\mathrm{d}y)^\infty = \mathscr{S}(\mathbb{R}^{2k})$$

(resp. H.S.$^\infty = \pi_\omega(l_{\mathscr{S}})$*).* ∎

Proof. Since the operators $\mathrm{d}R_y(X)$ (resp. $\mathrm{d}\overline{R}_x(X)$) generate the ring of all differential operators in $\dfrac{\partial}{\partial y_j}$ $\left(\text{resp. } \dfrac{\partial}{\partial x_j}\right)$ with polynomial coefficients ([11]) the operators $\mathrm{d}\widetilde{R}(X, Y)$ generate the ring of all differential operators with polynomial coefficients in the variables $(x,y) \in \mathbb{R}^{2k}$. Then the first assumption of the proposition is a consequence of the results of [13].

If $u \in \mathscr{S}(W)$, then $\pi_\omega(l_u) \in$ H.S. ([7]) and $\pi_\omega(l_u)\ \psi(v) = \psi(u * v)$ for any finite linear combination v of the Ω_l's.

In the same conditions, the infinitesimal action of R is given by:

$$\mathrm{d}RX^-(\pi_\omega(l_u)\ \psi(v)) = X^-(\pi_\omega(l_u)\ \psi(v)) + i\gamma(X,\cdot)(\pi_\omega(l_u)\ \psi(v)); \quad X \in g$$

(using notations of (49')). From lemma (3.7) we deduce that: $\mathrm{d}R(X) \circ \pi_\omega(l_u)) = = \pi_\omega(l_{P*u})$ where P is a polynomial function in (p,q). Similarly:

$$\pi_\omega(l_u) \circ \mathrm{d}R(X) = \pi_\omega(l_{u*P}).$$

Then $\pi_\omega(l_u)$ is a C^∞ vector for $\hat{R}$.

Conversely if $A \in HS^\infty$ and if δ is the element of the enveloping algebra $\mathscr{U}(g)$ of g such that $\mathrm{d}R(\delta_x) = \sum_{j=1}^{k} -\dfrac{\partial^2}{\partial x_j^2} + x_j^2$, then: $\mathrm{d}\hat{R}(\delta_x^n \cdot \delta_y^{n'})(A) \in$ H.S. ($\delta_x^n \cdot \delta_y^{n'}$ is the element of $\mathscr{U}(g+g) = \mathscr{U}(g) \otimes \mathscr{U}(g)$ product of $\delta^n \otimes 1$ by $1 \otimes \delta^{n'}$) and for any integers n, n' we have:

$$(\mathrm{d}\hat{R}(\delta_x^n \cdot \delta_y^{n'}) A k_l \mid k_m) = (\mathrm{d}R(\delta_x^n) A\, \mathrm{d}R(\delta_y^{n'}) k_l \mid k_m) =$$
$$= (A\, \mathrm{d}R(\delta_y^{n'}) k_l \mid \mathrm{d}R(\delta_x^n) k_m) \qquad (k_l \in L^2(\mathbb{R}^k, \mathrm{d}x)^\infty)$$
$$= (2k + |l| + 2)^{n'} (2k + |m| + 2)^n (A k_l \mid k_m).$$

Then if we write:

$$A k_l = \sum_m C_{l,m} k_m \quad \text{with} \quad C_{l,m} = (A k_l \mid k_m) \quad \text{then:}$$

$$\sum_{l,m} |C_{l,m}|^2 (2k + |l| + 2)^{2n'} (2k + |m| + 2)^{2n} < \infty; \quad n, n' \in \mathbb{N}. \tag{54}$$

$u = \sum_{l,m} C_{l,m}\, \Omega_m * \overline{\Omega}_l$ belongs to $L^2(W, \mathrm{d}p\, \mathrm{d}q)$ and moreover thanks to (54) and lemma 3.5, for any polynomial function P and Q in (p, q), $P *_\lambda u *_\lambda Q$ belongs to $L^2(W, \mathrm{d}p\, \mathrm{d}q)$. We shall prove that in fact $u \in \mathscr{S}(W)$ and $\pi_\omega(l_u) = A$.

Since

$$p_j(\Omega_l * \overline{\Omega}_m) = \frac{1}{2} (p_j *_\lambda \Omega_l * \overline{\Omega}_m + \Omega_l * \overline{\Omega}_m *_\lambda p_j)$$

$$\frac{\partial}{\partial p_j} (\Omega_l * \overline{\Omega}_m) = \frac{1}{2i} (q_j *_\lambda \Omega_l * \overline{\Omega}_m - \Omega_l * \overline{\Omega}_m *_\lambda q_j)$$

and similar expression for $q_j(\Omega_l * \overline{\Omega}_m)$ and $\dfrac{\partial}{\partial q_j} (\Omega_l * \overline{\Omega}_m)$, any differential operator D with polynomial coefficients acts on u on the form $Du = \sum_i P_i *_\lambda u *_\lambda Q_i \in L^2(W, \mathrm{d}p\, \mathrm{d}q)$. Then u is a C^∞ vector for the representation of the $2k$-dimensional Heisenberg group given at the infinitesimal level by $\left(ip_j, \dfrac{\partial}{\partial q_j}, iq_j, \dfrac{\partial}{\partial p_j}, j = 1, \ldots, k\right)$ i.e. $u \in \mathscr{S}(W)$. Now for fixed multiindices l_0, m_0 we have:

$$(\pi_\omega(l_u) k_{l_0}, k_{m_0}) = (\psi(u * \Omega_{l_0}) \mid \psi(\Omega_{m_0})) =$$
$$= (u * \Omega_{l_0} \mid \Omega_{m_0}) =$$
$$= \sum_{m,l} C_{l,m} (\Omega_m * \overline{\Omega}_l, \Omega_{m_0} * \overline{\Omega}_{l_0}) = C_{l_0, m_0}.$$

Therefore $\pi_\omega(l_u) = A$. ■

THEOREM 3.4. *Let* (p, q) *and* (p', q') *be two adapted charts on* W *and*

$(\mathfrak{G}(p,q),\tau)$ $(\mathfrak{G}(p',q'),\tau')$ the associated Von Neuman algebras and representations. Then, there exists a spatial isomorphism S from $\mathfrak{G}(p,q)$ onto $\mathfrak{G}(p',q')$ such that:

$$\tau'_g = S \circ \tau_g \circ S^{-1}; \qquad g \in G. \qquad \blacksquare$$

With the new adapted chart, we define a new $*$-product $*'$, a new algebra $\mathfrak{G}(p',q') = \overline{l'_{\mathscr{S}}}$ and a state ω' etc. . . . .

In $\mathscr{H}_{\omega'}$, we denote by R' the unitary representation of G. Following Kirillov there exists a unitary operator V from $\mathscr{H}_\omega$ to $\mathscr{H}_{\omega'}$ such that:

$$V \circ R \circ V^{-1} = R'.$$

See [14] for an explicit expression of V.

Cleraly the operator $\hat{V}$ from H.S. to H.S.' defined by:

$$V(A) = V \circ A \circ V^{-1}$$

induces an unitary equivalence between $\hat{R}$ and $\hat{R}'$. Let u be an element in $\mathscr{S}(W)$, since $A = \pi_\omega(l_u)$ is C^∞ for $\hat{R}$ (prop. 3.5), $\hat{V}(A)$ is C^∞ for $\hat{R}'$ i.e. in H.S.$^\infty$, thus there exists an unique element $w(u)$ in $\mathscr{S}(W)$ such that:

$$\hat{V}(\pi_\omega(l_u)) = \pi_{\omega'}(l'_{w(u)}). \tag{55}$$

The relations:

$$\begin{aligned}\hat{V}(\pi_\omega(l_{u_1})) \circ \hat{V}(\pi_\omega(l_{u_2})) &= \pi_{\omega'}(l'_{w(u_1)})\, \pi_{\omega'}(l'_{w(u_2)}) = \\ &= \pi_{\omega'}(l'_{w(u_1) *' w(u_2)}) = \\ &= \hat{V}(\pi_\omega(l_{u_1}) \circ \pi_\omega(l_{u_2})) = \\ &= \hat{V}(\pi_\omega(l_{u_1 * u_2}))\end{aligned}$$

and:

$$\hat{V}(\pi_\omega(l_u)^*) = \hat{V}(\pi_\omega(l_{\bar{u}})) = \pi_{\omega'}(l_{w(u)})^* = \pi_{\omega'}(l_{\overline{w(u)}})$$

imply:

$$w(u_1 * u_2) = w(u_1) *' w(u_2); \quad w(\bar{u}) = \overline{w(u)}.$$

w is unitary since:

$$(\pi_\omega(l_u)\, k_l \mid k_m) = (u * \Omega_l \mid \Omega_m) = (u \mid \Omega_m * \bar{\Omega}_l); \quad l, m \in \mathbb{N}^k$$

implies:

$$\| \pi_\omega(l_u) \|_{\text{H.S.}} = \| u \|_{L^2} = \| \pi_{\omega'}(l_{v(u)}) \|_{\text{H.S.}'} = \| w(u) \|_{L^2}$$

and the same relation holds for w^{-1}.

We extend w and w^{-1} to L^2 without change of notations. Finally:

$$\begin{aligned}\pi_{\omega'}(\tau'_g(l_{w(u)})) &= R'(g)\,\pi_{\omega'}(l_{w(u)})\,R'(g^{-1}) = \\ &= R'(g)\,\hat{V}(\pi_\omega(l_u))\,R'(g^{-1}) = \\ &= V(R(g)\,\pi_\omega(l_u)\,R(g^{-1}))\,V^{-1} = \\ &= \hat{V}(\pi_\omega(\tau_g(l_u))) = \hat{V}(\pi_\omega(l_{U(g)u})) = \\ &= \pi_{\omega'}(l_{w(U(g)u)}) = \pi_{\omega'}(l_{U'(g)w(u)}).\end{aligned}$$

Let us define S from $\mathfrak{G}\,(p,q)$ to $\mathfrak{G}\,(p',q')$ by:

$$S(T) = w \circ T \circ w^{-1}$$

we have a spatial isomorphism such that:

$$\begin{aligned}S(\tau_g l_u) = S(l_{U(g)u}) = l_{w(U(g)u)} = l_{U'(g)w(u)} = \tau'_g(l_{w(u)}) = \\ = \tau'_g S(l_u).\end{aligned}$$

Q.E.D.

Remark 3.5. Generally, w is distinct from the unitary operator between $L^2(W, \mathrm{d}p\,\mathrm{d}q)$ and $L^2(W, \mathrm{d}p'\,\mathrm{d}q')$ defined with the change of variables. Thus we see that the $*$-products defined on $\mathscr{S}(W)$ by means of an adapted chart depend of this chart. However theorem 3.4 proves are all unitarily equivalent. ∎

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